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Time Evolution of Infinitely Many Particles: An Existence Theorem

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In this paper we deal with systems of infinitely many particles in $\mathbb{R}^3$, given by a two-body, short-range potential and an external potential, depending on the position of the particles. We show the existence of dynamics for a set of initial configurations, which has measure one with respect to the Gibbs measure induced by a suitable family of Hamiltonians.

KEY WORDS: Systems of infinitely many particles; Gibbs measure; spatial perturbations of the equilibrium.

1. INTRODUCTION

The existence of time evolution of infinitely many classical particles has been much investigated in the recent literature. Such systems are very difficult to study, since the presence of infinitely many particles could give rise to catastrophic situations, which actually do happen: a particle with infinite velocity or infinite particles in a bounded region in a finite time.

This problem arises quite naturally in a statistical mechanical framework[(1)] and a reasonable way to approach it is to prove that, for a sufficiently large set of initial conditions, the motion of one particle is not so much affected by very far away particles. Then the evolution of each particle is essentially described by what happens in a bounded region around it, and this enables us to define the infinite dynamics as the limit of suitable finite particles dynamics.

Much care has to be devoted to the set of initial configurations which we want to evolve. It would be very easy to make the evolution of some

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statistically "rare" set of initial data (e.g., with finite energy), but this would be uninterestıng from a statistical mechanical point of view. Actually, we are interested in describing the evolution of a set of initial conditions, sufficiently large to be the support of thermodynamically interesting measures, like the Gibbs states.

In 1968 Lanford[2] proved an existence theorem for the dynamics of one-dimensional systems given by a two-body, short-range, and smooth potential. Dobrushin and Fritz in 1977 enlarged his results to singular potentials[3] and to two dimensions.[4] Moreover, they showed, by constructing an example, that their arguments, based on the energy conservation law, would not work at dimensions higher than two.

A very recent result is given by Pulvirenti.[5] He studies the evolution of states, instead of phase points, and shows the existence of the dynamics induced by a Hamiltonian H_0, for a state which is Gibbs with respect to $H_0 + \hat{h}$, where $\hat{h}(q)$ is an external potential satisfying

$$\int_{\mathbb{R}^d} \exp\left[-\hat{h}(q)\right]\left\{\exp\left[\frac{\partial \hat{h}(q)}{\partial q}\right]^2 - 1\right\} dq < +\infty \tag{1.1}$$

where $d \geqslant 1$ is the physical dimension.

In this paper we try to make further progress in this direction. We show the existence of the solution of the equations of motion for an explicit set of initial configurations $\mathscr{X}_0$, which is of full measure with respect to a Gibbs measure ν, given by a Hamiltonian $\hat{H} = H_0 + \hat{h}$; H_0 is given by both kinetic energy and a two-body, smooth, and short-range potential, and $\hat{h}$ is an external potential which forces the particles to be "quasi-one-dimensionally" distributed. More precisely, $\hat{h}$ must satisfy the following condition:

$$\int_{\Gamma} e^{-\alpha \hat{h}(q)}\, dq \leqslant c_\alpha \sigma^{1+\epsilon} \qquad \forall \alpha \in \mathbb{R}^+ \tag{1.2}$$

for any sphere Γ of diameter σ and $\epsilon \leqslant -2 + \sqrt{5}$ (c_α is a positive constant depending only on α). All configurations in $\mathscr{X}_0$ evolve under the action of a Hamiltonian $H = H_0 + h$, where h is any external, smooth, one-body potential, bounded from below, with $h(q) \leqslant \hat{h}(q)$.

We prove the existence of infinite dynamics for such systems, by means of a method used in Ref. 6 and classical energy conservation arguments.

We remark that the techniques involved here are completely different from those used in Ref. 5. Also, condition (1.2) is unrelated to (1.1), unless we are in the presence of asymptotically divergent external fields: in this case, (1.2) is weaker than (1.1).

2. DEFINITIONS, NOTATIONS, AND RESULTS

Let us consider a system of infinitely many particles of unitary mass in $\mathbb{R}^3$. We recall some usual definitions, which we will make use of in the following.

Let x denote a double sequence of positions and velocities, i.e., $x = \{(q_i, p_i)\}_{i=1\ldots\infty}$, $(q_i, p_i) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Definition 1.1. x is called a *locally finite configuration* if for any Λ bounded open set of $\mathbb{R}^3$ it is:

$$|x \cap (\Lambda \times \mathbb{R}^3)| < +\infty$$

where

$$x \cap (\Lambda \times \mathbb{R}^3) = \{(q_i, p_i) \in x : q_i \in \Lambda\}$$

and $|\{\cdot\}|$ stands for the cardinality of the set $\{\cdot\}$.

Let $\mathscr{X}$, the *infinite phase space*, be the set of all locally finite configurations, after having identified all sequences which differ only by a permutation. $\mathscr{X}$ is a topological space, if we define the following convergence:

Definition 2.2: Local Convergence. A sequence $x^n = \{(q_i^n, p_i^n)\}_{i=1,\ldots,\infty}$ converges to $x = \{(q_i, p_i)\}_{i=1\ldots\infty}$ in $\mathscr{X}$ if (i) for every Λ bounded open set of $\mathbb{R}^3$ such that $|x \cap (\partial\Lambda \times \mathbb{R}^3)| = 0$, it is definitively

$$|x^n \cap (\Lambda \times \mathbb{R}^3)| = |x \cap (\Lambda \times \mathbb{R}^3)|$$

(ii) there is an ordering on particles in Λ such that $\lim_{n\to\infty} p_i^n = p_i$ and $\lim_{n\to\infty} q_i^n = q_i$.

Let Σ denote the σ algebra of Borel sets of $\mathscr{X}$.

Definition 2.3. A *state* is a probability measure on $(\mathscr{X}, \Sigma)$.

For any $\Lambda \subset \mathbb{R}^3$, let $\mathscr{X}(\Lambda)$ be the phase space of all finite configurations, namely,

$$\mathscr{X}(\Lambda) = \bigcup_{n \geqslant 0} \mathscr{X}_n(\Lambda)$$

where $\mathscr{X}_n(\Lambda)$ is the symmetrization of $(\Lambda \times \mathbb{R}^3)^n$. $\mathscr{X}_n(\Lambda)$ is endowed with its natural topology.

Definition 2.4. A *finite volume state* ν_Λ is a probability measure on $(\mathscr{X}(\Lambda), \Sigma(\Lambda))$, where $\Sigma(\Lambda)$ is the σ algebra of Borel sets in $\mathscr{X}(\Lambda)$.

Our particles interact via a two-body potential Φ: $\mathbb{R}^+ \to \mathbb{R}$, $r \to \Phi(r)$.

It is given also the action of an external potential $h:\mathbb{R}^3\to\mathbb{R}$, $q\to h(q)$. We assume the following:

Properties 2.5. (a) *Short range*: $\Phi(r)=0$, $\forall r\geqslant R>0$. (b) $\Phi\in\mathcal{C}^2([0,R],\mathbb{R})$, $\Phi'(0)=0$. (c) *Stability*: Let

$$U(q_1,\ldots,q_n)=\sum_{\substack{i,j=1,\ldots,n\\ j\neq i}}\frac{1}{2}\Phi(\|q_i-q_j\|)$$

then there is a positive constant B_1 such that

$$U(q_1,\ldots,q_n)\geqslant -B_1 n$$

(d) $h\in\mathcal{C}^2(\mathbb{R}^3,\mathbb{R})$ and there exist b, c_k, a_k, α' positive constants, with $k=1,2$ and $\alpha'<1$ such that

(i) $$h(q)\geqslant -b,\qquad \forall q\in\mathbb{R}^3$$

(ii) $$|\mathfrak{D}^{(k)}h(q)|\leqslant c_k[|h(q)|]^{\alpha'}+a_k$$

where

$$|\mathfrak{D}^{(1)}h(q)|=\max_{r=1,2,3}|(\nabla h(q))_r|$$

$$|\mathfrak{D}^{(2)}h(q)|=\max_{r,s=1,2,3}\left|\frac{\partial(\nabla h(q))_r}{\partial(q)_s}\right|$$

with $(v)_k$ the kth component of the vector v.

We want to show that a considerably large set of initial configurations $\mathcal{X}_0\subset\mathcal{X}$ evolves following the Newton equations. More precisely, we want to prove the existence of a one-parameter group of transformations of $\mathcal{X}_0$ into itself, T_t: $x\to x(t)=T_t x$ such that $x(0)=x$ and, setting $x(t)=\{q_i(t), p_i(t)\}_{i=1,\ldots,\infty}$, it is

$$\begin{cases}\dot q_i=p_i\\ \dot p_i=F_i(T_t x)\end{cases}\tag{2.1}$$

where $F_i(T_t x)=-\sum_{i\neq j}\nabla_i\Phi(\|q_i-q_j\|)-\nabla_i h(q_i)$.

The flow T_t will be obtained as limit of suitable partial dynamics, which we now define.

For any $t\in\mathbb{R}$, $x\in\mathcal{X}$, and Λ_n open ball of $\mathbb{R}^3$ centered at the origin and of diameter n, let $T_t^n x=x^n(t)$ be the solution at time t of the Newton equations describing the evolution of $x\cap(\Lambda_n\times\mathbb{R}^3)$, induced by both interparticle and external fields, assuming that particles in Λ_n do not interact with those in Λ_n^c, which are frozen. For the hypotheses on h and Φ and for the local finiteness of x, such solution does exist for any $t\in\mathbb{R}$.

T_t^n is called the nth partial dynamics.

Given $\mu\in\mathbb{R}^3$, $\sigma\in\mathbb{R}^+$, let $\Gamma_\sigma(\mu)$ denote an open sphere in $\mathbb{R}^3$, of diameter σ and center μ.

For any $x \in \mathfrak{X}$ let us define the following quantity:

$$H(x;\ \mu, \sigma) = \overline{\sum_i} \frac{1}{2} \|p_i\|^2 + \frac{\Lambda}{2} \overline{\sum_{\substack{i,j \\ i \neq j}}} \Phi(\|q_i - q_j\|) + \overline{\sum_i} h(q_i) + \overline{\sum_i} (B+1) \tag{2.2}$$

where $\overline{\sum}$ means that the sum is taken over the particles in $\Gamma_\sigma(\mu)$ and $B = B_1 + b$.

Notice that, by Properties 2.5 (c), (d), $H(x;\ \mu, \sigma) > 0$.

For any ϵ satisfying $0 < \epsilon < \sqrt{5} - 2$, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function defined as

$$\begin{aligned} &\text{(a)} \qquad \varphi(\cdot) = \max\{1, (\cdot)^\gamma\} \\ &\text{(b)} \qquad (1+\epsilon)/2 \leqslant \gamma < (1-\epsilon)/(1+\epsilon) \end{aligned} \tag{2.3}$$

Let us define

$$Q(x) = \sup_{\mu} \sup_{\sigma \geqslant \varphi(\|\mu\|)} \frac{H(x;\ \mu, \sigma)}{\sigma^{1+\epsilon}} \tag{2.4}$$

$$\mathfrak{X}_0 = \{x \in \mathfrak{X} : Q(x) < +\infty\} \tag{2.5}$$

We will show the existence of the evolution for configurations in $\mathfrak{X}_0$. Furthermore this set will turn out to be of measure 1 with respect to physically "interesting" measures.

We give now our main theorem, whose proof will follow in Section 3.

Theorem 2.6. Suppose Φ and h satisfy Properties 2.5, with $\alpha' < (1-\epsilon)/(1+\epsilon)$, ϵ as in (2.3). It is possible to define a one-parameter group of transformations $T_t : \mathfrak{X}_0 \to \mathfrak{X}_0\ x \to T_t x$ such that the following are true:

(i) $T_t x = \lim_{n\to\infty} T_t^n x$ in the sense of the local convergence, for any $t \in \mathbb{R}$, $x \in \mathfrak{X}_0$.

(ii) If we put $T_t x = \{q_i(t), p_i(t)\}_{i=1,\ldots,\infty}$, then for any i, q_i and p_i are differentiable functions of time and satisfy (2.1).

(iii) There exist a continuous increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\omega \in \mathbb{R}^+$ such that

$$Q(T_t x) \leqslant g(|t|)[Q(x)]^\omega$$

(iv) $T_t x$ is the only solution of (2.1) in the class of all solutions satisfying the bound:

$$Q(T_t x) \leqslant G(t, Q(x))$$

where G is any continuous function of both the arguments.

3. EXISTENCE AND UNIQUENESS OF THE DYNAMICS

Before going on with the proof of the theorem, we prove some estimates; we will make an essential use of the energy conservation law.

For any $x \in \mathcal{X}_0$, let us set

$$T_t^n x = \{q_i^n(t), p_i^n(t)\}$$

and

$$D(n,t) = \sup_{q_i(0) \in \Lambda_n} \sup_{0 \leqslant \tau \leqslant t} \|q_i^n(\tau) - q_i(0)\|$$

From now on we will write x in place of $x(0)$. Furthermore, Λ and Γ will stand for open spheres in $\mathbb{R}^3$ and any other region will be indicated with Ω.

Estimate 3.1.

$$D(n,t) \leqslant \sup_{q_i(0) \in \Lambda_n} \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau \|p_i^n(s)\| ds$$

$$\leqslant t\left[2H(x;0,n)\right]^{1/2} \leqslant t\left[2Q(x)\right]^{1/2} n^{(1+\epsilon)/2}.$$

This immediately follows from (2.4).

Let $\mathfrak{N}_i(n,t)$ be the number of particles that interact with the ith particle, in the nth dynamics, during the time interval $[0,t]$. We have

Estimate 3.2. There exists a positive constant C_0 such that, if $\|q_i(0)\| < n$,

$$\mathfrak{N}_i(n,t) \leqslant C_0 t^{1+\epsilon} Q(x)^{(3+\epsilon)/2} n^{(1+\epsilon)\gamma} \qquad [\gamma \text{ as in } (2.3)]$$

Proof. From (2.2), (2.4), and the properties of Φ, we get

$$\mathfrak{N}_i(n,t) \leqslant H(x; q_i(0), 2D(n,t) + R)$$

$$\leqslant Q(x) \cdot \left[2D(n,t) + R + \varphi(\|q_i(0)\|)\right]^{1+\epsilon}$$

The Estimate follows from 3.1 and (2.3).

From Property 2.5 (d) we finally have the following:

Estimate 3.3.

$$\sup_{q_i(0) \in \Lambda_n} \sup_{0 \leqslant \tau \leqslant t} |\mathfrak{D}^{(k)} h(q_i^n(\tau))| \leqslant c_k'\left[H(x;0,n)\right]^{\alpha'} \leqslant c_k'\left[Q(x)\right]^{\alpha'} n^{(1+\epsilon)\alpha'}$$

for $k = 1, 2$.

Proof of Theorem 2.6. We will give the proof in three steps. Step 1: for any $x \in \mathcal{X}_0$, $T_t^n x$ locally converges to $T_t x$; step 2: $T_t x$ is a one-parameter group of maps of $\mathcal{X}_0$ onto itself; step 3: $T_t x$ is unique in the sense of (iv).

Step 1. Let x be any configuration in $\mathcal{X}_0$ and $T_t^n x$ its evolution under the nth partial dynamics. Let Λ_{k_0} be an open sphere of $\mathbb{R}^3$ centered at the origin and of diameter k_0 and take $n \geqslant [k_0 + 2D(n,t) + R + 1]$, where $[\cdot]$ stands for the integer part.

We will denote

$$\delta_i(n,t) = \|q_i^{n+1}(t) - q_i^n(t)\|$$
$$u_k(n,t) = \sup_{q_i(0)\in\Lambda_k} \delta_i(n,t), \qquad k \leqslant n$$

Let n be so large that particles in Λ_{k_0}, in the $(n+1)$th partial dynamics, during the time interval $[0,t]$, are not influenced by particles initially in $\Lambda_{n+1}\backslash\Lambda_n$. Then by the Newton equations, we get

$$\begin{aligned}\delta_i(n,t) &\leqslant \int_0^t \|p_i^{n+1}(\tau) - p_i^n(\tau)\| \\ &\leqslant \int_0^t d\tau \int_0^\tau ds\, \|F_i(T_s^{n+1}x) - F_i(T_s^n x)\| \end{aligned} \tag{3.1}$$

Let us compute now

$$\begin{aligned}&\|F_i(T_s^{n+1}x) - F_i(T_s^n x)\| \\ &\quad\leqslant \sum_{i\neq j}\left\{|\nabla_i\Phi(\|q_i^{n+1}(s) - q_j^{n+1}(s)\|) - \nabla_i\Phi(\|q_i^n(s) - q_j^n(s)\|)\right\} \\ &\qquad + \|\nabla_i h(q_i^{n+1}(s)) - \nabla_i h(q_i^n(s))\| \\ &\quad\leqslant M\sum_{i\neq j}\left\{\left[\|q_j^{n+1}(s) - q_j^n(s)\| + \|q_i^{n+1}(s) - q_i^n(s)\|\right]\chi_i(q_j(s))\right\} \\ &\qquad + C_2 n^{(1+\epsilon)\alpha'}\|q_i^{n+1}(s) - q_i^n(s)\| \end{aligned} \tag{3.2}$$

this follows from Estimate 3.3, with $M = \max_{0\leqslant r\leqslant R}|\Phi''(r)|$ and χ_i the indicator function of the set $\{q_j : \|q_i(s) - q_j(s)\| \leqslant R\}$. Then, if we call Λ_{K_1} the open sphere containing the particles that could influence particles at time 0 in Λ_{k_0}, during the time interval $[0,t]$, we get

$$(3.2) \leqslant 2M\mathfrak{N}_i(n+1,t)u_{k_1}(n,t) + c_2 n^{(1+\epsilon)\alpha'}u_{k_1}(n,t)$$

then, by virtue of Estimate 3.2, taking the supremum over Λ_{k_0} in (3.1), we have

$$u_{k_0}(n,t) \leqslant C_3[\,Q(x)]^{(3+\epsilon)/2}t^{1+\epsilon}\left[n^{(1+\epsilon)\gamma} + n^{(1+\epsilon)\alpha'}\right]\int_0^t d\tau\int_0^\tau ds\cdot u_{k_1}(n,s)$$

and $k_1 = [k_0 + 2D(n,t) + R + 1]$

We can iterate the procedure l times, if

$$l(2D(n,t) + R) < n/4 \tag{3.3}$$

From Estimate 3.1, choosing $l = C_4\{[\,Q(x)]^{1/2}t\}^{-1}n^{(1-\epsilon)/2}$ inequality (3.3) is satisfied, and hence

$$u_{k_0}(n,t) \leqslant \left\{2C_3\left[Q(x)\right]^{1/2}t\right\}^{(3+\epsilon)l} n^{(1+\epsilon)(l+1)\delta} \cdot \left[(2l)!\right]^{-1} \tag{3.4}$$

with $\delta = \max\{\alpha', \gamma\}$. Moreover, by the choice of l,

$$u_{k_0}(n,t) \leqslant C_5^l\left[Q(x)t^2\right]^{\delta' l} l^{2\delta l(1+\epsilon)/(1-\epsilon)} \cdot \left[(2l)!\right]^{-1} \tag{3.5}$$

with $\delta' = 1 + 2\delta/(1-\epsilon)$. Then $u_{k_0}(n,t) \to 0$ as $n \to \infty$ in an integrable way, having chosen ϵ and δ as in (2.3).

We can find the same result for the following quantity:

$$v_{k_0}(n,t) = \sup_{q_i(0) \in \Lambda_{k_0}} \eta_i(n,t)$$

with $\eta_i(n,t) = \|p_i^{n+1}(t) - p_i^n(t)\|$.

Analogously we get

$$v_{k_0}(n,t) \leqslant C_6^{l+1}\left[Q(x)t^2\right]^{\delta'(l+1)} l^{2\delta(1+\epsilon)(l+1)/(1-\epsilon)}\left[(2l)!\right]^{-1} \tag{3.6}$$

(C_3, C_4, C_5, C_6 suitable positive constants).

This shows that $q_i^n(t) \to q_i(t)$ and $p_i^n(t) \to P_i(t)$ as $n \to \infty$.

Call $T_i x = \{q_i(t), p_i(t)\}_{i=1,\ldots,\infty}$. In order to prove that the convergence is in $\mathcal{X}$, it remains to be shown that $T_t x$ is locally finite. To this purpose, we now prove an n-uniform estimate for the displacements. If $s < n$, we can write

$$\|q_i^n(t) - q_i(0)\| \leqslant \|q_i^s(t) - q_i(0)\| + \sum_{m=s}^{n-1} \|q_i^{m+1}(t) - q_i^m(t)\| \tag{3.7}$$

Again let $q_i(0) \in \Lambda_{k_0}$ and

$$s \geqslant 2\left[K_0 + 2D(s,t) + R\right] \tag{3.8}$$

By Estimate 3.1 and Eqs. (3.3) and (3.4) we have

$$\|q_i^n(t) - q_i(0)\| \leqslant t\left[2Q(x)\right]^{1/2} s^{(1+\epsilon)/2} + b_1(x,t) \sum_{m \geqslant s} m^{-\theta(Q(x),t)m^\nu} \tag{3.9}$$

where

$$0 < \theta \leqslant \mathrm{const}/t\left[Q(x)\right]^{1/2}, \ \nu = (1-\epsilon)/2, \quad \text{and} \quad b_1 = \left\{t\left[2Q(x)\right]^{1/2}\right\}^2.$$

Then

$$\begin{aligned} D_{\Lambda_{k_0}}(n,t) &= \sup_{q_i(0) \in \Lambda_{k_0}} \sup_{0 \leqslant \tau \leqslant t} \|q_i^n(\tau) - q_i(0)\| \\ &\leqslant t\left[2Q(x)\right]^{1/2} s^{(1+\epsilon)/2} + b_1(x,t) e^{-\theta(Q(x),t)s^\nu} \\ &\leqslant 2\left\{t\left[Q(x)\right]^{1/2}\right\}^2 s^{(1+\epsilon)/2} \end{aligned} \tag{3.10}$$

By (3.8) and Estimate 3.1, there exist b_2 and b_3 positive constants such that s can be chosen as

$$s = \left\{\left[Q(x)\right]^{1/2} t\right\}^{\xi}(b_2 + b_3 k_0), \qquad \xi \in \mathbb{R}^+ \tag{3.11}$$

Then, by (3.10) and (3.11) it turns out that the number of particles of $T_t^n x$ in $\Lambda_{k_0}, N_{\Lambda_{k_0}}, (T_t^n x)$, satisfies

$$\begin{aligned} N_{\Lambda_{k_0}}(T_t^n x) &\leqslant Q(x)\left[K_0 + 2b_1(x,t) s^{2(1+\epsilon)}\right]^{1+\epsilon} \\ &\leqslant b_4 Q(x)\left\{ t\left[Q(x)\right]^{1/2}\right\}^{b_5(1+\epsilon)^2} k_0^{1+\epsilon} \end{aligned} \tag{3.12}$$

Moreover, let us note that, if Ω_σ is any bounded open region of $\mathbb{R}^3$ with diameter σ, we have $\Omega_\sigma \subseteq \Gamma_{\sigma'}(\mu) \subseteq \Lambda_{k_0}$ for some μ where $k_0 = 2(\|\mu\| + \sigma')$ and σ' is some multiple of σ. Then, analogously we get

$$N_{\Omega_\sigma}(T_t^n x) \leqslant b_\sigma Q(x)\left\{ t\left[Q(x)\right]^{1/2}\right\}^{b_5(1+\epsilon)^2} (\|\mu\| + \sigma)^{1+\epsilon} \tag{3.13}$$

b_4, b_5, b_6 are positive constants. This means that $T_t x$ is locally finite. This, together with (3.5) and (3.6) concludes the proof of step 1.

Step 2. Let $x \in \mathfrak{X}_0$ and $\Gamma_\sigma(\mu)$ such that

$$|T_t x \cap (\partial\Gamma_\sigma(\mu) \times \mathbb{R}^3)| = 0$$

Then for $n \geqslant N$ sufficiently large, it is:

$$|T_t^n x \cap (\partial\Gamma_\sigma(\mu) \times \mathbb{R}^3)| = 0$$

and furthermore

$$|T_t^n x \cap (\Gamma_\sigma(\mu) \times \mathbb{R}^3)| = |T_t x \cap (\Gamma_\sigma(\mu) \times \mathbb{R}^3)|$$

We shall prove that it is possible to find an n-uniform estimate for $H(T_t^n x;\ \mu, \sigma)$. This will allow us to define

$$Q(T_t x) = \sup_{\mu \in \mathbb{R}^3} \ \sup_{\sigma > \varphi(\|\mu\|)} \frac{H(T_t x;\ \mu, \sigma)}{\sigma^{1+\epsilon}} < +\infty \tag{3.14}$$

(3.14) will be proven, once we succeed in showing that

$$Q(T_t x) \leqslant g(|t|)\left[Q(x)\right]^{\omega} \tag{3.15}$$

which is a condition stronger than (3.14).

Let $s \geqslant N$; we have

$$H(T_t^n x;\ \mu, \sigma) = H(T_t^s x;\ \mu, \sigma) + \sum_{m=s}^{n-1} H(T_t^{m+1} x;\ \mu, \sigma) - H(T_t^m x;\ \mu, \sigma) \tag{3.16}$$

Let us call

$$\mathcal{K}_\Gamma(t,m) = H(T_t^{m+1}x;\ \mu,\sigma) - H(T_t^m x;\ \mu,\sigma) \tag{3.17}$$

[Γ stands for $\Gamma_\sigma(\mu)$]. It is

$$\begin{aligned}|\mathcal{K}_\Gamma(t,m)| \leqslant{}& N_\Gamma(T_t^m x)\big[2Q(x)\big]^{1/2} m^{(1+\epsilon)/2} \sup_{q_i(0)\in\Gamma} \eta_i(m,t)\\ &+ N_\Gamma^2(T_t^m x) M \sup_{q_i(0)\in\Gamma} \delta_i(m,t)\\ &+ N_\Gamma(T_t^m x) \sup_{q_i(0)\in\Gamma} \delta_i(m,t) C_3 m^{(1+\epsilon)\alpha}\end{aligned} \tag{3.18}$$

Now we choose s as in (3.11), with $k_0 = \|\mu\| + \sigma$. Using (3.5), (3.6), and (3.13) we get

$$|\mathcal{K}_\Gamma(t,m)| \leqslant a_\Lambda(t,x)(\|\mu\| + \sigma)^{2(1+\epsilon)} m^{(1+\epsilon)\beta'} e^{-\theta(t,x)m^\nu} \tag{3.19}$$

where $\beta' = \max\{\alpha', \frac{1}{2}\}$ and

$$a_1(t,x) = g(|t|)[\,Q(x)]^\omega \tag{3.20}$$

with $\omega \in \mathbb{R}^+$ and $g(|t|) = \text{const}\cdot|t|^z$, $z \in \mathbb{R}^+$. Therefore

$$\left|\sum_{m=s}^{n-1} \mathcal{K}_\Gamma(t,m)\right| \leqslant g(|t|)[\,Q(x)]^\omega (\|\mu\| + \sigma)^{2(1+\epsilon)} s^{(1+\epsilon)\beta'} e^{-\theta s^\nu} \tag{3.21}$$

Since s is proportional to $\|\mu\| + \sigma$, the right hand side of (2.21) goes to 0 as $s \to \infty$; then, for sufficiently large s,

$$\left|\sum_{m=s}^{n-1} \mathcal{K}_\Gamma(t,m)\right| \leqslant a_2 g(|t|)[\,Q(x)]^\omega \tag{3.22}$$

(a_2 is a positive constant). Finally, (3.11) and Estimate 3.1 imply

$$\begin{aligned}H(T_t^s x;\ \mu,\sigma) &\leqslant H(x;\ \mu, \sigma + 2D_\Gamma(s,t))\\ &\leqslant Q(x)\left[\left(\sigma + \left\{t[\,Q(x)]^{1/2}\right\}^{(1+\epsilon)(\xi/2+1)} (\sigma + \|\mu\|)^{(1+\epsilon)/2}\right)\right]^{1+\epsilon}\end{aligned} \tag{3.23}$$

Then (3.15) follows by (3.22), (3.23) and (2.3). The same result holds also for any bounded open region for which $|T_t^n x \cap (\partial\Omega \times \mathbb{R}^3)| \neq 0$. In fact, since x is locally finite, it is possible to find an open sphere $\Gamma_\sigma(\mu)$ which strictly contains Ω and such that

$$|T_t^n x \cap (\partial\Gamma_\sigma(\mu) \times \mathbb{R}^3)| = 0 \tag{3.24}$$

Step 3. Uniqueness of solutions is immediately seen, by the same arguments used in step 1. If $q_i(t)$ and $Q_i(t)$ denote position at time t

corresponding to the same initial datum $q_i(0) = Q_i(0)$, we have

$$\|q_i(t) - Q_i(t)\| \leqslant \int_0^t \|F_i(q_i(\tau)) - F_i(Q_i(\tau))|d\tau$$
$$\leqslant \text{const} \cdot \|q_i(0)\|^{(1+\epsilon)\delta} \int_0^t d\tau \int_0^\tau ds \sup_{q_j \in \Lambda_{k_1}} \|q_j(s) - Q_j(s)\| \tag{3.25}$$

where

$$k_1 = \|q_i(0)\| + 2t[Q(x)]^{1/2}\|q_i(0)\|^{(1+\epsilon)/2}$$

We can iterate the procedure n times, with n arbitrarily large:

$$(3.25) \leqslant \text{const}^n \cdot \left(\|q_i(0)\| + n\|q_i(0)\|^{(1+\epsilon)/2}\right)^{(1+\epsilon)\sigma n} \cdot \left[(2n)!\right]^{-1} \tag{3.26}$$

But the right-hand side of this inequality runs to 0 as $n \to \infty$ and so we get the uniqueness.

4. $\mathfrak{X}_0$ IS A FULL MEASURE SET

In Section 3 we have constructed a one-parameter group of transformations defined on a subset $\mathfrak{X}_0 \subset \mathfrak{X}$. Our purpose is now to prove that $\mathfrak{X}_0$ is a measure-1 set, with respect to the Gibbs measure induced by a class of suitable Hamiltonians.

Let us introduce the following definition:

Definition 4.1 (Superstability). Denote by $\mathcal{P}_2$ a partition of $\mathbb{R}^3$ with cubes Δ of side $\lambda > R$ and by $n(x, \Delta)$ the number of particles of x in Δ. Let $U(x) = \sum_{i \neq j} \frac{1}{2}\Phi(\|q_i - q_j\|)$. U will be called superstable if there exist positive constants A and B such that

$$U(x) \geqslant \sum_{\Delta \in \mathcal{P}_\lambda} An^2(x, \Delta) - Bn(x, \Delta)$$

Consider the following function on $\mathfrak{X}$:

$$\hat{H}(x) = \sum_i \tfrac{1}{2}\|p_i\|^2 + \sum_{i \neq j} \tfrac{1}{2}\Phi(\|q_i - q_j\|) + \sum_i \hat{h}(q_i)$$

where Φ satisfies Properties 2.5 (a), (b) and Definition 4.1, and $\hat{h}$ satisfies the following properties:

Properties 4.2. (a) Let $\alpha > 0$; there exists $C_\alpha > 0$ such that, for any open sphere $\Gamma_\sigma(\mu)$ it is

$$\int_\Gamma e^{-\alpha\hat{h}(q_i)}\, dq_i \leqslant C_\alpha \sigma^{1+\epsilon}$$

(b) $\hat{h}(q) - h(q) \geqslant 0$ where h is the external potential in (2.2).

Define the following measure:

$$\lambda(dx) = 1 + \sum_{n>0} \frac{1}{n!} dq_1\, dp_1 \ldots dq_n\, dp_n$$

with $dq_i\, dp_i$ the Lebesgue measure in $\mathbb{R}^3 \times \mathbb{R}^3$. A finite *volume Gibbs measure* relative to the Hamiltonian H is the probability measure on $(\mathcal{X}(\Lambda), \Sigma(\Lambda))$

$$\nu_{\beta,\Lambda}(dx) = \frac{1}{Z_\Lambda} e^{-\beta \hat{H}(x)} \lambda(dx) \tag{4.1}$$

where Z_Λ, the normalization, is given by

$$Z_\Lambda = \int_{\mathcal{X}(\Lambda)} e^{-\beta \hat{H}(x)} \lambda(dx) \tag{4.2}$$

with $\beta = 1/kT$, T is the temperature, and k the Boltzmann constant; Λ is any bounded Borel set in $\mathcal{X}$.

Equation (4.1) is well posed for the properties of $U(x)$ and $\hat{h}(q)$.

Let us recall another definition. Let Λ be a bounded Borel set in $\mathcal{X}$ and $q_1, \ldots, q_n$ n particles in Λ. A *finite volume correlation function* relative to the Hamiltonian H is the following function:

$$\begin{aligned} &\rho_\Lambda(q_1, \ldots, q_n; p_1, \ldots, p_n) \\ &\quad = \frac{1}{z_\Lambda} \sum_{k>0} \frac{1}{k!} \\ &\qquad \times \int_{(\Lambda \times \mathbb{R}^3)^k} \exp\left(-\beta \left\{ U(q_1, \ldots, q_{n+k}) + \sum_{i=1}^{n+k} \left[\hat{h}(q_i) + \frac{\|p_i\|^2}{2} \right] \right\} \right) \\ &\qquad\quad dq_1 \ldots dq_k\, dp_1 \ldots dp_k \end{aligned} \tag{4.3}$$

$\rho_\Lambda(q_1, \ldots, q_n; p_1, \ldots, p_n)$ is proportional to the probability measure (with respect to the Lebesgue measure) of finding n particles in the positions $q_1, \ldots, q_n$ in Λ.

We will give now a crucial estimate for the correlation functions. After this, we shall get existence for the infinite volume Gibbs measure and a control on the expectation value of H on any open sphere.

First of all, we need a theorem whose proof, rather technical, which is a slight modification of that given in Ref. 7, will be found in the Appendix. In this proof, we make essentially use of the superstability of $U(x)$.

Theorem 4.3. Let Φ satisfy Properties 2.5 and Definition 4.1. There are $\theta, \delta > 0$, $\theta < 1$ such that

$$\begin{aligned} &\rho_\Lambda(q_1, \ldots, q_n; p_1, \ldots, p_n) \\ &\quad \leqslant \exp\left\{ -\beta\theta U(q_1, \ldots, q_n) - \beta \sum_{i=1}^{n} \left[\frac{\|p_i\|^2}{2} + \hat{h}(q_i) \right] + n\delta \right\} \end{aligned} \tag{4.4}$$

For any $\Sigma(\Lambda)$-measurable function f, denote by

$$\nu_{\beta,\Lambda}(f) = \int_{\mathscr{X}(\Lambda)} f(x)\nu_{\beta,\Lambda}(dx) \tag{4.5}$$

the expectation value of f relative to $\nu_{\beta,\Lambda}$.

Theorem 4.4. Let $\nu_{\beta,\Lambda}$ the Gibbs measure defined in (4.1) and $\Gamma_\sigma(\mu) \subset \Lambda$. For $\lambda > 0$ sufficiently small, there exists a $\xi > 0$ such that

$$\nu_{\beta,\Lambda}(e^{\lambda H(x;\mu,\sigma)}) \leqslant e^{\xi_\sigma 1+\epsilon}$$

Proof. From Definition (4.3) we get

$$\begin{aligned}
&\nu_{\beta,\Lambda}(e^{\lambda H(x;\mu,\sigma)}) \\
&\quad \leqslant \sum_{k\geqslant 0} \frac{1}{k!}\int_{(\Gamma\times\mathbb{R}^3)^k} \rho_\Lambda(q_1,\ldots,q_k,p_1,\ldots,p_k) \\
&\qquad \times \exp\Bigg(\lambda\Bigg[U(q_1,\ldots,q_k) \\
&\qquad\qquad + \sum_{i=1}^{k}\left\{\frac{\|p_i\|^2}{2} + h(q_i) + 2(B+1)\right\}\Bigg]\Bigg) dq_1\,dp_1\ldots dq_k\,dp_k
\end{aligned}$$

By Theorem 4.3 and Property 4.2(b), choosing λ so small that $\beta\theta - \lambda \geqslant 0$ we get

$$\nu_{\beta,\Lambda}(e^{\lambda H(x;\mu,\sigma)}) \leqslant \sum_{k\geqslant 0}\frac{z'^k}{k!}e^{\delta' k}\left(\int_\Gamma e^{-(\beta-\lambda)\hat{h}(q_i)}\,dq_i\right)^k$$

where $z' = \int_{\mathbb{R}^3}\exp[-(\beta-\lambda)(\|p_i\|^2/2)\,dp_i]$ and $\delta' = \delta + (\beta\theta-\lambda)B + 2\lambda(B+1)$. Therefore, by Property 4.2(a), we get the thesis, with $\xi = z'e^{\delta'}\cdot$ const.

From Theorem 4.3 and Ref. 7, it is possible to define an infinite volume probability measure ν_β on $(\mathscr{X},\Sigma)$, as weak limit of finite volume Gibbs measures $\nu_{\beta,\Lambda}$, for some sequence of Borel bounded sets Λ invading $\mathbb{R}^3$, i.e.,

$$\nu_\beta(f) = \lim_{\Lambda\to\mathbb{R}^3}\nu_{\beta,\Lambda}(f) \tag{4.6}$$

for every continuous, bounded function $f\colon \mathscr{X}\to\mathbb{R}$. We prove the following theorem:

Theorem 4.5. Let ν_β be the infinite volume Gibbs probability measure defined in (4.6). Then

$$\nu_\beta(\mathscr{X}_0) = 1$$

Proof. First of all, we note that $\mathscr{X}_0 \in \Sigma$, since $Q(x)$ is a measurable function. We will get the thesis, if we can show that

$$\nu_\beta(\mathscr{X}_0^c) = 0$$

For any $s \in \mathbb{Z}^+$, let us set

$$D_s = \{\sigma \in \mathbb{R}^+ : \sigma = m/2^s, m \in \mathbb{Z}^+\}$$

and

$$\tilde{D}_s = \left\{ \mu \in \mathbb{R}^3, \mu \equiv (\mu_1, \mu_2, \mu_3) : \mu_i = m/2^s, m \in \mathbb{Z}\right\}$$

Then it is

$$\mathfrak{X}_0^c \subseteq \bigcup_{s=0}^{\infty} \{x : Q_s(x) = \infty\} = \bigcup_{s=0}^{\infty} \bigcap_{r=0}^{\infty} \{x : Q_s(x) > r\} \tag{4.7}$$

where

$$Q_s(x) = \sup_{\mu \in \tilde{D}_s} \sup_{\substack{\sigma \in D_s \\ \sigma \geqslant \varphi(\|\mu\|)}} \frac{H(x;\, \mu, \sigma)}{\sigma^{1+\epsilon}}$$

On the other hand, for any $s \in \mathbb{Z}^+$, we have

$$\{x : Q_s(x) > r\} \subseteq \bigcup_{\mu \in \tilde{D}_s} \bigcup_{\sigma \in D_s, \sigma \geqslant \varphi(\|\mu\|)} \left\{x : H(x;\, \mu, \sigma) > r\sigma^{1+\epsilon}\right\} \tag{4.8}$$

and so

$$\begin{aligned} &\nu_\beta(\{x : Q_s(x) > r\}) \\ &\quad \leqslant \sum_{k \geqslant 0} \left[(2k)^3 + 1\right] \sum_{\substack{\sigma \in D_s \\ \sigma \geqslant \varphi(\|\mu\|)}} \nu_\beta\left(\left\{x : e^{\lambda H(x;\mu,\sigma)} \geqslant e^{\lambda r \sigma^{1+\epsilon}}\right\}\right) \end{aligned} \tag{4.9}$$

with $\|\mu\| = \sup_i |\mu_i| = k/2^s$.

By Theorem 4.5 and the Tchebychev inequality it follows that for any $\Gamma_\sigma(\mu) \subset \Lambda$,

$$\begin{aligned} &\sum_{\sigma \geqslant \varphi(k/2^s)} \nu_{\beta,\Lambda}\left(\left\{x : \exp\left[\lambda H(x;\, \mu, \sigma)\right] > \exp(\lambda r \sigma^{1+\epsilon})\right\}\right) \\ &\qquad \leqslant \exp\left\{-(\lambda r - \xi)\left[\varphi(k/2^s)\right]^{1+\epsilon}\right\} \cdot \text{const} \end{aligned} \tag{4.10}$$

(4.10) is true also for ν_β, since it is independent of the choice of Λ. Then, for $r > 0$ large enough, we have

$$\nu_\beta(\{x : Q_s(x) > r\}) \leqslant \sum_{k \geqslant 0} \exp\left\{-(\lambda r - \xi)\left[\varphi(k/2^s)\right]^{1+\epsilon}\right\} \cdot \text{const} \tag{4.11}$$

and, for the properties of φ,

$$\nu_\beta(\{x : Q_s(x) > r\}) \to 0 \qquad \text{as } r \to \infty \tag{4.12}$$

So we get the thesis, by (4.7).

Remark. In this proof it is not necessary to take $\varphi(k)$ growing as k^γ; it would have been enough a logarithmic growth.

5. COMMENTS

Let us interpret the results that we have obtained from a physical point of view. Let $h = 0$ and $\hat{h}$ satisfying Property 4.2 then Theorem 4.4 proves that we are able to evolve a subset $\mathcal{X}_0$ of initial data such that $\nu_\beta(\mathcal{X}_0) = 1$ and to obtain an evolution for a class of states which are spatial perturbations of equilibrium states, Gibbs with respect to $\hat{H}$, of the system.

Note that $\mathcal{X}_0$ is a measure-1 set also for other classes of states. For example, we can consider states which exhibit suitable temperature gradients, together with an external field $\hat{h}$; $\mathcal{X}_0$ will still be of measure 1, as can be easily seen by obvious modifications in the proof of Theorem 4.5. If, on the contrary, we are in presence of a physical situation in which there is an external field h (for example, particles in a suitable electric field), then the previous result for $\hat{h} = h$ enables us to get the time evolution of nonequilibrium states supported on $\mathcal{X}_0$.

In this paper we have given a generalization of these two cases.

Let us also remark that Property 4.2(a) permits fields with cylindric symmetry, asymptotically growing at least as $(\log \rho)^{1+\eta}$, where $\eta > 0$ and ρ is the distance of the particle from the symmetry axis. Nevertheless, as we can see by a direct inspection of the arguments used in the proofs, it is possible to consider external fields growing as $\log \rho$. In this case, however, not every value for the temperature is allowed.

Finally, as a consequence of Theorem 2.6(iii), let us note that there exist time-dependent correlation functions, related to the time-evoluted Gibbs measures $\nu_{\beta,t}$ defined as

$$\nu_{\beta,t}(A) = \nu_\beta(T_{-t}A), \qquad \forall A \in \Sigma$$

In fact, for any bounded function f: $\Omega \to \mathbb{R}$, with $\Omega \subset (\mathbb{R}^3 \times \mathbb{R}^3)^n$ a compact set, there exist $\alpha \in \mathbb{R}^+$ such that

$$\begin{aligned}\int_{\mathcal{X}} \nu_{\beta,t}(dx)|f(x)| &\leqslant \max_{x \in \Omega} |f(x)| \cdot \text{cost.} \int \nu_{\beta,t}(dx)\left[N_\Omega(x)\right]^\alpha \\ &\leqslant \max_{x \in \Omega} |f(x)| \cdot \text{cost.}\ g(|t|)^\alpha \int_{\mathcal{X}} \nu_\beta(dx)\left[Q(x)\right]^{\omega\alpha}\end{aligned}$$

Then, by standard arguments, we can prove the existence of time-dependent correlation functions ρ_t, satisfying the BBKGY hierarchy in the weak form (e.g., see Ref. 5).

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APPENDIX: SOME PROBABILITY ESTIMATES

We will now give the proof of Theorem 4.3. We remark that our estimate holds also for divergent two-body potentials and slightly improves Ruelle's estimates.[7] Nevertheless the techniques involved are the same. Furthermore, $\hat{h}$ is not needed to satisfy Properties 4.2, since we will only use its boundedness property.

We note first that, in case of zero particles, Theorem 4.3 is immediately proven; then we shall make use of the inductive method on the number of particles. The result will be valid in $\mathbb{R}^d$, $d \geqslant 1$.

Let us define the following quantities.

Definition 4.1 (Interaction Energy).

$$W(x_\Lambda \mid x_{\Lambda'}) = \sum_{q_i \in \Lambda} \sum_{q_j \in \Lambda'} \Phi(\|q_i - q_j\|)$$

where

$$x_\Lambda = x \cap (\Lambda \times \mathbb{R}^d), \qquad x \in \mathcal{X}, \qquad \Lambda \subset \mathbb{R}^d, \qquad \Lambda' \subseteq \Lambda^c$$

Definition 4.2.

$$E_\Omega(x) = \sum_{\substack{\Delta_i \in \mathcal{P}_\lambda \\ \Delta_i \subset \Omega}} n^2(x, \Delta_i)$$

where $\mathcal{P}_\lambda$ and Δ_i are the same as in Definition 4.1 and $\Omega \subset \mathbb{R}^d$.

Definition 4.1 implies that, given any $\Lambda \subset \mathbb{R}^d$, we have the following decomposition:

$$U(x) = U(x_\Lambda) + U(x_{\Lambda^c}) + W(x_\Lambda \mid x_{\Lambda^c})$$

Consider now a sequence of open cubes in $\mathbb{R}^d$ of side $2kR$, $k \in \mathbb{Z}$ and R the range of Φ. From 4.1, 4.2 we immediately get the following Estimates:

Estimate 4.3. For any $k \geqslant 1$ and $x \in \mathcal{X}$ it is

$$W(x_{\Lambda_k} \mid x_{\Lambda_k^c}) \geqslant -m \sum_{\substack{\Delta_i \subset \Lambda_k \\ \Delta_j \subset \Lambda_{k+1} \setminus \Lambda_k}}' n(x, \Delta_i) n(x, \Delta_j) \geqslant -2^d m E_{\Lambda_{k+1}}(x)$$

Estimate 4.4. For any $k \geqslant 1$ and $x \in \mathcal{X}$ it is

$$W(x_{\Lambda_k} \mid x_{\Lambda_k^c}) \geqslant -m \sum_{\substack{\Delta_i \subset \Lambda_k \\ \Delta_j \subset \Lambda_{k+1} \setminus \Lambda_k}}' \frac{n^2(x, \Delta_i) + n^2(x, \Delta_j)}{2}$$

$$\geqslant -2^d m \left[E_{\Lambda_{k+1}}(x) - E_{\Lambda_{k-1}}(x) \right]$$

where $m = -\inf_{[0,R]} \Phi(r)$ and $\sum'$ is the sum over the Δ_i, Δ_j's first neighbors.

For any bounded open region $\Lambda \subset \mathbb{R}^d$ and $q > 1$, let us define the following sets:

$$\mathbb{S}_q = \{x \in \mathfrak{X}(\Lambda) : E_{\Lambda_{q-1}}(x) > a|\Lambda_{q-1}|\}$$
$$\tilde{\mathbb{S}}_q = \{x \in \mathfrak{X}(\Lambda) : E_{\Lambda_{q+j}}(x) \leqslant a|\Lambda_{q+j}|, \forall j \geqslant 0\}$$

Here $|\Lambda_q|$ stands for the volume of Λ_q and a is a positive constant.

Let χ_q and $\tilde{\chi}_q$ be the indicator functions on $\mathbb{S}_q$, $\tilde{\mathbb{S}}_q$, respectively. It is easily seen that

$$\mathfrak{X}(\Lambda) = \tilde{\mathbb{S}}_p \cup \left(\bigcap_{q>p} (\mathbb{S}_q \cap \tilde{\mathbb{S}}_q) \right) \tag{A1}$$

Moreover, let us put

$$I_p = \frac{1}{z_\Lambda} \int_{\mathfrak{X}(\Lambda)} \tilde{\chi}_p e^{-\beta \hat{H}(x \cup y)} \lambda(dy)$$
$$I_q = \frac{1}{z_\Lambda} \int_{\mathfrak{X}(\Lambda)} \chi_q \tilde{\chi}_q e^{-\beta \hat{H}(x \cup y)} \lambda(dy)$$

where $x = \{(q_i, p_i)\}_{i=1,\ldots,n}$. We can suppose, without loss of generality, that one of the n particles $q_1, \ldots, q_n$ is in the origin of $\mathbb{R}^d$. This will ensure that at least one particle is in Λ_p, $p \geqslant 1$. We will prove the following inequalities:

$$I_p \leqslant C_p \exp\left\{ -\beta\theta_1[U(x_{\Lambda_p}) + W(x_{\Lambda_p} \mid x_{\Lambda_p^c})] - \beta \sum_{q_i \in \Lambda_p} [\hat{h}(q_i) + \|p_i\|^2/z] + \delta_1 n(x, \Lambda_p) \right\} \rho_\Lambda(x_{\Lambda_p^c}) \tag{A2}$$

$$I_q \leqslant C_q \exp\left\{ -\beta\theta_2[U(x_{\Lambda_q}) + W(x_{\Lambda_q} \mid x_{\Lambda_q^c})] - \beta \sum_{q_i \in \Lambda_q} \left[\hat{h}(q_i) + \frac{\|p_i\|^2}{2} \right] + \delta_2 n(x, \Lambda_q) \right\} \rho_\Lambda(x_{\Lambda_q^c}) \tag{A3}$$

with θ_1, θ_2, δ_1, δ_2 positive constants; C_p and C_q positive constants depending on p, q, respectively, and such that

$$\sum_{q \geqslant 0} C_q < +\infty \tag{A4}$$

On the other hand, we have

$$\rho_\Lambda(q_1, \ldots, q_n; p_1, \ldots, p_n) = I_p + \sum_{q>p} I_q \tag{A5}$$

Therefore, since $|x \cap \Lambda_k^c| \leqslant n$, $\forall k \geqslant p$, we can apply induction on the right-hand side A2, A3, and then, by (A5), get the thesis.

Proof of (A2). Let $z = x \cup y$, $y = \{(q_i', p_i')\} \in \zeta_p$. By stability of U and Estimates 4.3, for $\theta_1 < 1$ we get

$$\begin{aligned} U(z) \geqslant \theta_1\big[U(x_{\Lambda_p}) + W(x_{\Lambda_p}|X_{\Lambda_p^c})\big] + U(z_{\Lambda_p^c}) - Bh(q,\Lambda_p) \\ - (1-\theta_1)Bh(x,\Lambda_p) - 2^d ma\big[(1+\theta_1)|\Lambda_{p+1}| + |\Lambda_p|\big] \end{aligned} \tag{A6}$$

and then, by the decomposition $\lambda(dy) = \lambda(dy_{\Lambda_p}) \cdot \lambda(dy_{\Lambda_p^c})$

$$\begin{aligned} I_p \leqslant \exp\Big\{ & -\beta\theta_1\big[U(x_{\Lambda_p}) + W(x_{\Lambda_p}|x_{\Lambda_p^c})\big] \\ & -\beta \sum_{q_i \in \Lambda_p} \big[\hat{h}(q_i) + \|p_i\|^2/2\big] + \delta_1 n(x,\Lambda_p)\Big\} \\ & \times \int \lambda(dy_{\Lambda_p}) \exp\Big\{ Bn(y,\Lambda_p) - \beta \sum_{q_i' \in \Lambda_p^c} \big[\hat{h}(q_i') + \|p_i'\|^2/2\big]\Big\} \\ & \times 1/z_\Lambda \int \lambda(dy_{\Lambda_p^c}) \exp\Big\{ -\beta U(z_{\Lambda_p^c}) - \beta \sum_{q_i' \in \Lambda_p^c} \Big[\hat{h}(q_i') + \frac{\|p_i'\|^2}{2}\Big] \\ & + \beta m 2^d a(2+\theta_1) \cdot |\Lambda_{p+1}|\Big\} \end{aligned} \tag{A7}$$

and then we get (A2), with

$$c_p = \exp\big[\beta m 2^d C_1 |\Lambda p + 1| + z e^{\beta(B+b)} C |\Lambda_p|\big]$$

Proof of (A3). Let $z = x \cup y$, $y \in \zeta_q \cap \tilde{\zeta}_q$. By Estimates 4.4 and superstability of U, for $\theta_2 < B/(2^{d+2}m + B)$, we have

$$\begin{aligned} U(z_{\Lambda_q}) \geqslant\ & \theta_2\big[U(z_{\Lambda_q}) + W(z_{\Lambda_q} \,|\, z_{\Lambda_q^c})\big] \\ & + (1-\theta_2)\big[U(z_{\Lambda_q}) + W(z_{\Lambda_q} \,|\, z_{\Lambda_q^c})\big] + U(z_{\Lambda_q^c}) \\ \geqslant\ & U(z_{\Lambda_q^c}) + \theta_2\big[U(x_{\Lambda_q}) + W(x_{\Lambda_q} \,|\, x_{\Lambda_q^c})\big] + \Big(C_3 + \frac{1-\theta_2}{2} C_2\Big)|\Lambda_q| \\ & - m(1+\theta_2)C_4 q^{d-1} - \frac{1+\theta_2}{2} Bn(y,\Lambda_q) - \frac{(1-\theta_2)}{2} Bn(x,\Lambda_q) \end{aligned} \tag{A8}$$

Then

$$I_q \leqslant \exp\Big\{ -\beta\theta_2\big[U(x_{\Lambda_q}) + W(x_{\Lambda_q} \,|\, x_{\Lambda_q^c})\big] + \delta_2 n(x, \Lambda_q) - \beta \sum_{q_i \in \Lambda_q} \big[\hat{h}(q_i) + \|p_i\|^2/2 \big] \Big\}$$
$$\times \int \lambda(dy_{\Lambda_q}) \exp\Big\{ \frac{\beta B}{2} \cdot \Big(1 + \frac{\theta}{2}\Big) n(y, \Lambda_q) - \beta \sum_{q_i' \in \Lambda_q} \Big[\hat{h}(q_i') + \frac{\|p_i'\|^2}{2} \Big] \Big\}$$
$$\times 1/Z_\Lambda \cdot \int \lambda(dy_{\Lambda_q^c}) \exp\Big\{ -\beta - U(z_{\Lambda_q^c}) - \beta \sum_{q_i' \in \Lambda_q^c} \big[\hat{h}(q_i') + \|p_i'\|^2/2 \big] \Big\}$$
$$\times \exp\big[-C_5|\Lambda_q| + m(1 + \theta_2) C_4 q^{d-1} \big] \tag{A9}$$

Let us set

$$C_q = \exp\big\{ z \exp\big[\beta B/2 \cdot (1 + \theta_2) + b\beta \big] q^d + C_6 q^{d-1} - C_7 q^d \big\} \tag{A10}$$

where C_7 is a positive constant proportional to a. Then (A4) follows, for sufficiently large a, and this implies (A3).

REFERENCES

1. O. E. Lanford, *Dynamical Systems*, J. Moser, ed. Lecture Notes in Physics No. 38 (Springer, Berlin, 1975), p. 1.
2. O. E. Lanford, *Commun. Math. Phys.* **9**:176 (1968).
3. R. L. Dobrushin and J. Fritz, *Commun. Math. Phys.* **55**:275 (1977).
4. J. Fritz and R. L. Dobrushin, *Commun. Math. Phys.* **57**:67 (1977).
5. M. Pulvirenti, *J. Stat. Phys.* (to appear).
6. C. Marchioro, A. Pellegrinotti, and M. Pulvirenti, Proceedings of Estergorm Summer School (1979) (to appear).
7. D. Ruelle, *Commun. Math. Phys.* **18**:127 (1970).